Score:

Name: Solutions

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

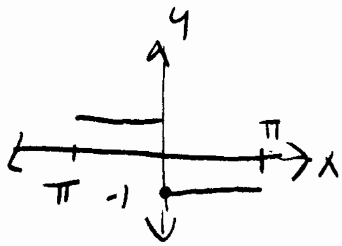
SM316 – Test #3– Fall 2006

Calculators/one note sheet/book allowed. Box/circle your final answer.
(Use no more that 2 decimals to answer all questions)
YOU MUST SHOW ALL WORK FOR FULL CREDIT.

1. (20 points) Find the Fourier series (not complex) for the following function:

$$f(x)=\begin{cases}-1 & 0\le x\le \pi \\ 1 & -\pi \le x \le 0\end{cases}$$

Determine the first 3 non-zero terms of the series. Sketch the series as the number of terms goes to infinity.



Function is odd so a_0 and a_n equal 0 (i.e. no cosine terms)

$$b_n = \underset{L}{\frac{1}{\pi}}\int_{-\pi}^{\pi} f(x)\sin(\overset{\frac{n\pi x}{L}}{nx})\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\sin(nx)dx + \int_{0}^{\pi}-\sin(nx)dx\right]$$

$$= \frac{-1}{n\pi}\cos(nx)\Big|_{-\pi}^{0} - \frac{-1}{n\pi}\cos nx\Big|_{0}^{\pi}$$

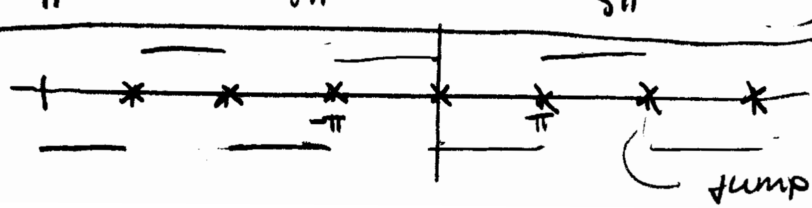
$$= \frac{-1}{n\pi}(1-\cos(-n\pi)) + \frac{1}{n\pi}(\cos(n\pi)-1)$$

$$= \frac{-1}{n\pi}(1-\cos(n\pi)) - \frac{1}{n\pi}(1-\cos(n\pi))$$

$$= \frac{-2}{n\pi}(1-(-1)^n) \Rightarrow b_n = \begin{cases}0 & n=\text{even}\\ -\frac{4}{n\pi} & n=\text{odd}\end{cases}$$

$$\boxed{f(x)\sim -\frac{4}{\pi}\sin(x) - \frac{4}{3\pi}\sin(3x) - \frac{4}{5\pi}\sin(5x)}$$

PLOT



jump discontinuities

No marks on this table		
1		
2		
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4. (20 points) Find the Fourier transform for the function in Problem #1. After calculating C_ω sketch the function given by $f(x) \sim \int_{-\infty}^{\infty} C_\omega e^{i\omega x} d\omega$.

$$C_\omega = +\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-\pi}^{0} e^{-i\omega x}\, dx - \int_{0}^{\pi} e^{-i\omega x}\, dx$$

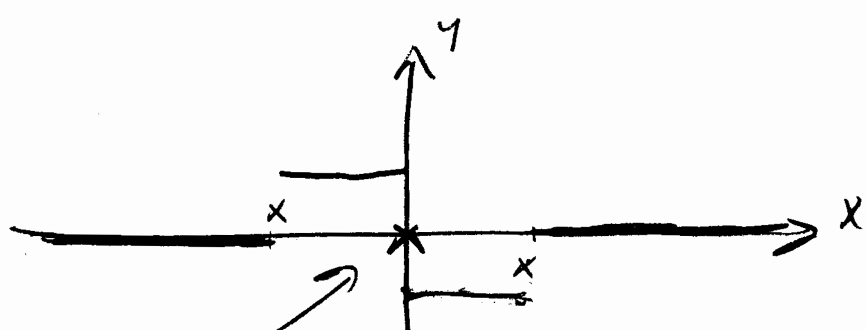
$$= -\frac{1}{i\omega} e^{-i\omega x}\Big|_{-\pi}^{0} + \frac{1}{i\omega} e^{-i\omega x}\Big|_{0}^{\pi}$$

$$= \frac{-1}{i\omega}\left(1 - e^{+i\omega\pi}\right) + \frac{1}{i\omega}\left(e^{-i\omega\pi} - 1\right)$$

$$= \frac{-2}{i\omega} + \frac{1}{i\omega} e^{i\omega\pi} + \frac{1}{i\omega} e^{-i\omega\pi}$$

$$= \frac{-2}{i\omega} + \frac{1}{i\omega}\left(\cos(\omega\pi) + \cancel{i\sin(\omega\pi)} + \cos(\omega\pi) - \cancel{i\sin(\omega\pi)}\right)$$

$$= \frac{2i}{\omega} - \frac{2i}{\omega}\cos(\omega\pi) = \boxed{\frac{2i}{\omega}\left(1 - \cos(\omega\pi)\right)}$$



jump discontinuities

2. (20 points) Find the complex Fourier series for the function in #1. Determine the $n = -3$ through $n = 3$ terms of the series. Sketch a frequency spectrum for the series.

$$d_0 = a_0 = 0 \qquad \omega_n = \frac{n\pi}{L} = n$$

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i\omega_n x}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}\,dx - \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\,dx$$

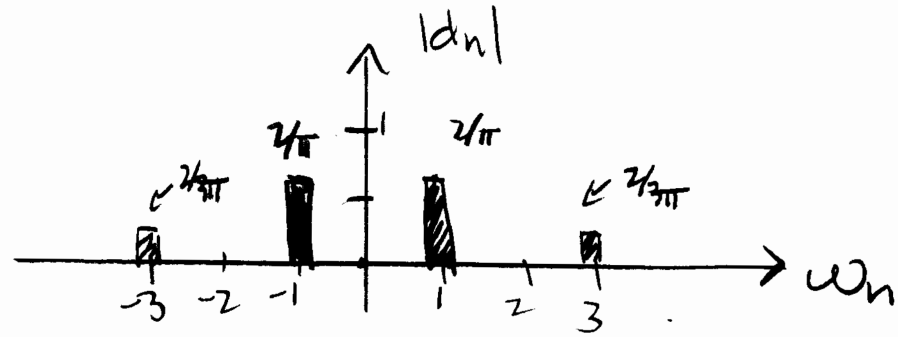
$$= \frac{-1}{2\pi i n} e^{-inx}\Big|_{-\pi}^{0} \overset{+}{-} \frac{-1}{2\pi i n} e^{-inx}\Big|_{0}^{\pi}$$

$$= \left(\frac{-1}{2\pi i n} + \frac{1}{2\pi i n} e^{-in\pi}\right) + \left(\frac{1}{2\pi i n} e^{-in\pi} - \frac{1}{2\pi i n}\right)$$

$$= -\frac{1}{n\pi i} + \frac{1}{n\pi i} e^{-in\pi} \Rightarrow d_n = \begin{cases} 0 & n \text{ even} \\ \frac{-2}{n\pi i} & n \text{ odd} \end{cases}$$

$$\left(\text{note } \frac{-2}{n\pi i}\frac{(i)}{(i)} = \frac{2i}{n\pi}\right)$$

$$\Rightarrow f(x) \sim \boxed{\frac{-2i}{3\pi} e^{-3ix} - \frac{2i}{\pi} e^{-ix} + \frac{2i}{\pi} e^{ix} + \frac{2i}{3\pi} e^{3ix}}$$



Frequency Spectrum

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3. (20 points) Show that the sum of the $n=-1$ and $n=1$ term in Problem #2 equals the term $n=1$ in Problem #1.

$$-\frac{2i}{\pi}e^{-ix} + \frac{2i}{\pi}e^{ix}$$

$$= -\frac{2i}{\pi}(\cancel{\cos(x)} - i\sin(x)) + \frac{2i}{\pi}(\cancel{\cos(x)} + i\sin(x))$$

$$= \frac{4i}{\pi}(i)\sin(x) = -\frac{4}{\pi}\sin(x)$$

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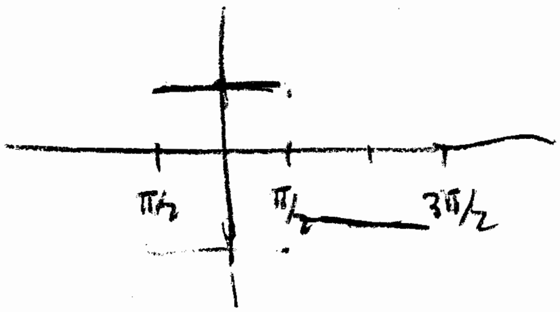
5. (20 points) Find the Fourier transform for the function: $f(x) = \begin{cases} -1 & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ 1 & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \end{cases}$.



This is just problem 4 Shifted Right by π/2

$$\Rightarrow \mathcal{F}\{f(t-\overset{\pi/2}{t_0})\} = e^{-it_0\omega}\,\mathcal{F}\{f(t)\}$$

$$\boxed{C_\omega = e^{-\frac{\pi}{2}i\omega}\left[\frac{2i}{\omega}\cdot(1-\cos(\omega\pi))\right]}$$