Score:

Name:
Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM316 – Quiz #10 (Section 14.1, 14.7) – In Class Exercise

Take home quiz, open book, open notes. You may work with team members to solve problems, but you may not copy another's work. Calculators are allowed, but you must show all work for full credit.

1. Determine the first five non-zero terms of the Fourier series for the following function:

$$f(x) = \pi - |x| \text{ for } -\pi \le x \le \pi \Rightarrow f(x) = \begin{cases} \pi - x & 0 \le x \le \pi \\ \pi + x & -\pi \le x \le 0 \end{cases}$$

Plot the series as the number of terms approaches infinite.

$\frac{n\pi x}{L} = nx$ for $L=\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} (\pi + x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) + \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - |x|)\cos(nx)\,dx$$

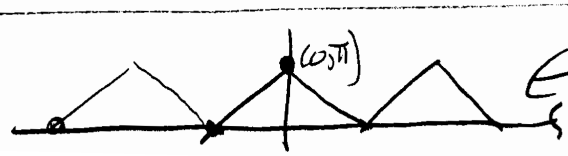
$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{\cos(n\pi)}{n^2}\right) + \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{\cos(n\pi)}{n^2}\right)$$

$$= \frac{2}{n^2\pi}\left(1 - (-1)^n\right) \Rightarrow \boxed{a_n = \begin{cases} 4/n^2\pi & n \text{ odd} \\ 0 & n \text{ even} \end{cases}}$$

$\Rightarrow b_n = 0$ since $\pi - |x|$ is even funtion

$$\boxed{f(x) \sim \frac{\pi}{2} + \frac{4}{\pi}\cos(x) + \frac{4}{9\pi}\cos(3x) + \frac{4}{25\pi}\cos(5x) + \frac{4}{49\pi}\cos(7x)}$$



← plot as $n \to \infty$

2. Determine the first five terms of the complex Fourier series $(-2 \le n \le 2)$ for the function in #1. Plot a frequency spectrum of the first five terms.

since $L=\pi$

$$f(x) \sim d_0 + \sum_{n=-\infty}^{\infty} d_n e^{i\frac{n\pi x}{L}} = d_0 + \sum_{-\infty}^{\infty} d_n e^{-inx}$$

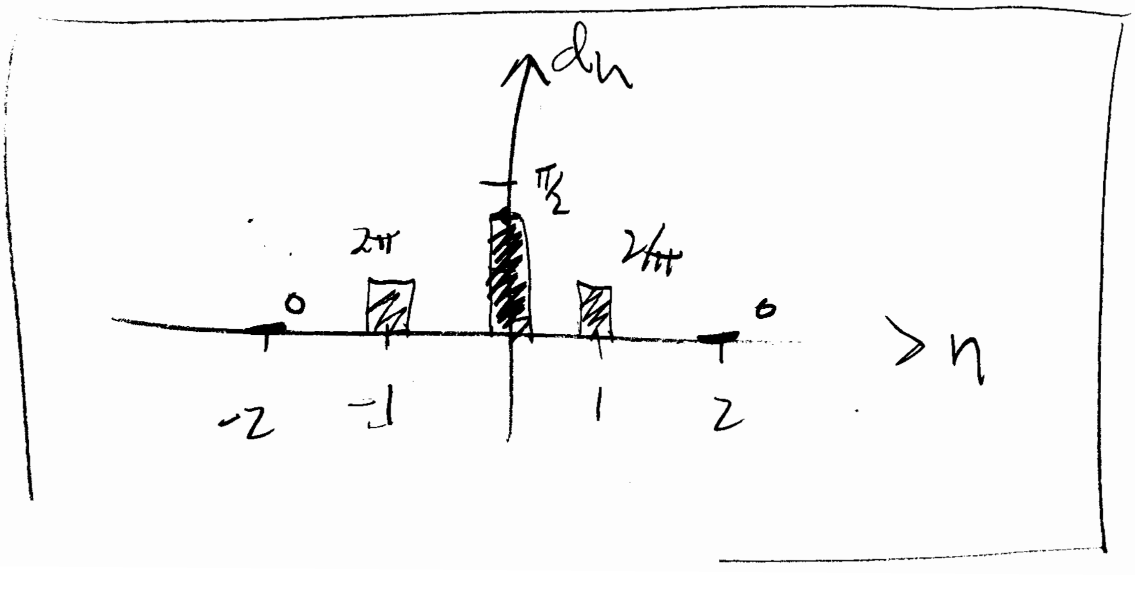
$$d_0 = a_0 = \pi/2$$

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} (\pi+x) e^{-inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} (\pi-x) e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{n^2}\left(1-(-1)^n\right) + \cancel{\frac{\pi}{n i}}\right) + \frac{1}{2\pi}\left(\frac{1}{n^2}\left(1-(-1)^n\right) - \cancel{\frac{\pi}{n i}}\right)$$

$$= \frac{1}{n^2\pi}\left(1-(-1)^n\right) \Rightarrow \boxed{d_n = \begin{cases} \frac{2}{n^2\pi} & n = \text{odd} \\ 0 & n \text{ even} \end{cases}}$$

$$\Rightarrow \boxed{f(x) \sim \overset{-2}{0} + \overset{-1}{\frac{2}{\pi} e^{-ix}} + \overset{0}{\frac{\pi}{2}} + \overset{1}{\frac{2}{\pi} e^{ix}} + \overset{2}{0}}$$



③ $f(x) = \begin{cases} k & -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow C_\omega = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-a}^{a} k e^{-i\omega t}\, dt = k \int_{-a}^{a} e^{-i\omega t}\, dt$$

$$= \frac{k}{-i\omega} e^{-i\omega t} \Big|_{-a}^{a} = \frac{-k}{i\omega}\left(e^{-i\omega a} - e^{i\omega a}\right)$$

$$= -\frac{k}{i\omega}\left(\cancel{\cos(\omega a)} - i\sin(\omega a) - (\cancel{\cos(\omega a)} + i\sin(\omega a))\right)$$

$$= -\frac{k}{\cancel{i}\omega}\left(-2\cancel{i}\sin(\omega a)\right) = \boxed{\frac{2k}{\omega}\sin(\omega a)}$$